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Journal of Geometry and Physics 51 (2004) 34–39

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

A characterization of totally umbilical hypersurfaces in de Sitter space

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Received 25 February 2003

Abstract

It is shown that a compact spacelike hypersurface contained in the chronological future (or past) of an equator of de Sitter space is a totally umbilical round sphere if there exist nonnegative constants $C_1, C_2, \dots, C_{l-1}$, at least one C_i is positive, such that

$$H_l = \sum_{i=1}^{l-1} C_i H_i.$$

This extends the previous result in [J. Geom. Phys. 39 (2001), Theorem 2].

MSC: 53C40; 53C42

JGP SC: General relativity; Differential geometry

Keywords: Mean curvature function; Minkowski formula; Umbilical hypersurface; Spacelike hypersurface; de Sitter space

1. Introduction

Let $\mathbb{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

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doi:10.1016/j.geomphys.2003.09.006

and let $\mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ be the $(n+1)$-dimensional unitary de Sitter space

$$\mathbb{S}_1^{n+1} = \{x \in \mathbb{L}^{n+2} : \langle x, x\rangle = 1\}.$$

It is well known that, for $n \geq 2$, the de Sitter space $\mathbb{S}_1^{n+1}$ is the standard simply connected Lorentzian space form of positive constant sectional curvature. A smooth immersion $\psi : M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ of a connected n-dimensional manifold M^n is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on M^n, which will also be denoted by $\langle\cdot,\cdot\rangle$. When M^n is compact without boundary, the following characterization of the totally umbilical round hypersurface is recently known (for Riemannian case, see [5,8,11]).

Theorem 1 ([4, Theorem 2]). *Let $\psi : M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space, which is contained in the chronological future* (*or past*) *of an equator of $\mathbb{S}_1^{n+1}$. If H_l does not vanish on M^n and the ratio H_k/H_l is constant for some k, l, $1 \leq l < k \leq n$, then M^n is a totally umbilical round sphere.*

In this note, we generalize Theorem 1 in the following way.

Theorem 2. *Let $\psi : M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space, which is contained in the chronological future* (*or past*) *of an equator of $\mathbb{S}_1^{n+1}$. If there are nonnegative constants $C_1, C_2, \ldots, C_{l-1}$, at least one C_i is positive, such that*

$$H_l = \sum_{i=1}^{l-1} C_i H_i,$$

then M^n is a totally umbilical round sphere.

The Riemannian version of Theorem 2 under the convexity assumption is proven in [10]. After the submission of this paper, we were informed that similar result was given in Theorem 6.1 of [3]. It should be remarked that, while our proof is simpler, Theorem 6.1 in [3] works for a very wide class of ambient spacetimes, not only for de Sitter space.

2. Preliminaries

Let M^n be a compact spacelike hypersurface in de Sitter space, then M^n is diffeomorphic to an n-sphere [2], hence is orientable. Then there exists a timelike unit normal field η globally defined on M^n. Now, let A be the shape operator of M^n in $\mathbb{S}_1^{n+1}$ with respect to η, which is given by

$$A(X) = -\mathrm{d}\eta(X).$$

Let $\kappa_1, \kappa_2, \ldots, \kappa_n$ be principal curvatures of M^n and let σ_k be the kth elementary symmetric polynomial of the principal curvatures:

$$\sigma_k(\kappa_1, \kappa_2, \ldots, \kappa_n) = \sum_{i_1 < i_2 < \cdots < i_k} \kappa_{i_1} \cdots \kappa_{i_k}, \quad 1 \leq k \leq n,$$

then kth mean curvature function H_k of the spacelike hypersurface is defined by

$$\binom{n}{k} H_k = (-1)^k \sigma_k(\kappa_1, \dots, \kappa_n) = \sigma_k(-\kappa_1, \dots, -\kappa_n).$$

Then $H_1 = -(1/n)\text{trace}(A)$ is the usual mean curvature function, H_2 is the scalar curvature of M^n up to a constant, $H_n = (-1)^n \det(A)$ is the Gauss–Kronecker curvature function and H_0 is defined to be $H_0 \equiv 1$.

The following Minkowski formula will be essential to our proof of Theorem 2.

Lemma 1 ([2, Theorem 2]). Let $\psi : M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ *be a compact spacelike hypersurface immersed into de Sitter space and let* $a \in \mathbb{L}^{n+1}$ *be an arbitrary fixed vector. For each* $r = 0, 1, \dots, n-1$, *the following identities hold*:

$$\int_M (-H_r\langle a, \psi\rangle + H_{r+1}\langle a, \eta\rangle)\, \mathrm{d}M = 0.$$

The following facts will also be used.

Lemma 2. *Let* $\psi : M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ *be the same as in* Lemma 1. *Suppose that* $H_l > 0$ *for some* $l = 2, 3, \dots, n$ *and that there exists a point* $p_0 \in M$ *where all the principal curvatures* $\kappa_1(p_0), \dots, \kappa_n(p_0)$ *are negative*.

(1) *For* $i \leq l$, $H_i > 0$. *Furthermore, it holds that*

$$\frac{H_i}{H_l} \geq \frac{H_{i-1}}{H_{l-1}},$$

and the equality holds if and only if all the principal curvatures are the same.

(2) *If there are nonnegative constants* $C_1, C_2, \dots, C_{l-1}$ *such that* $H_l = \sum_{i=1}^{l-1} C_i H_i$, *then it holds that* $H_{l-1} \geq \sum_{i=1}^{l-1} C_i H_{i-1}$, *and if, furthermore, the equality holds, then all the principal curvatures are the same*.

Proof.

(1) For the first inequality, see [9, Lemma 1]. For the second inequality, see, for example, [6].

(2) Dividing the first identity by H_l, we have from (1) that

$$1 = \sum_{i=1}^{l-1} C_i \frac{H_i}{H_l} \geq \sum_{i=1}^{l-1} C_i \frac{H_{i-1}}{H_{l-1}}$$

or

$$H_{l-1} - \sum_{i=1}^{l-1} C_i H_{i-1} \geq 0.$$

If the identity holds, it implies

$$\frac{H_i}{H_l} = \frac{H_{i-1}}{H_{l-1}},$$

which implies that all the principal curvatures are the same by (1). □

3. Proof

Let us assume, for instance, that the hypersurface $\psi : M^n \to \mathbb{S}_1^{n+1} \in \mathbb{L}^{n+2}$ is contained in the future of the equator determined by a unit timelike vector $a \in \mathbb{L}^{n+2}$ (the case of the past is similar). This means that $\psi(M^n) \subset \{x \in \mathbb{S}_1^{n+1} : \langle a, x\rangle < 0\}$, that is

$$\langle a, \psi\rangle < 0,$$

on M^n. We first show that $H_l > 0$ on M^n. Let us orient M^n by the globally defined timelike unit normal field η so that

$$\langle a, \eta\rangle \leq -1 < 0.$$

Since M^n is compact, there exists a point, say, p_0 at which the height function $\langle a, \psi\rangle$ attains its maximum. On the other hand, since $\langle a, \psi\rangle < 0$ on M^n, we have

$$\langle a, \psi(p_0)\rangle = \max_{p\in M} \langle a, \psi(p)\rangle < 0.$$

Furthermore, its Hessian at p_0, $\nabla^2\langle a, \psi\rangle(p_0)$ satisfies

$$\nabla^2\langle a, \psi\rangle(p_0)(v, w) = -\langle a, \psi(p_0)\rangle\langle v, w\rangle - \langle a, \eta(p_0)\rangle\langle A_{p_0}(v), w\rangle \leq 0$$

for all $v, w \in T_{p_0}M$. Now, take $v = w$ as a principal direction, then since

$$\langle v, v\rangle > 0, \qquad \langle a, \psi(p_0)\rangle < 0, \qquad \langle a, \eta(p_0)\rangle < 0,$$

we have

$$\langle A_{p_0}(v), v\rangle \leq -\frac{\langle a, \psi(p_0)\rangle}{\langle a, \eta(p_0)\rangle}\langle v, v\rangle < 0,$$

that is

$$\kappa_i(p_0) < 0, \quad l = 1, 2, \ldots, n,$$

in particular

$$H_l(p_0) > 0.$$

We now claim that $H_l(p) > 0$ for every point $p \in M$. The following proof of this claim is essentially the same as in [1], however, we include here for completeness. Let U be the connected component of the set $\{p \in M : H_l(p) > 0\}$ containing p_0. It is clear that U is a nonempty open subset of M. We will show that it is also closed. By Garding's inequality

[7] (taking into account the sign convention in the definition of H_l), we know that at each point $p \in U$

$$H_i^{l/i}(p) \geq H_l(p) > 0$$

for every $1 \leq i \leq l-1$. We also know that there exists at least one positive constant C_i, say $C_k > 0$. Then, at each point $p \in U$, we have

$$H_l(p) = \sum_{i=1}^{l-1} C_i H_i(p) \geq C_k H_k(p).$$

On the other hand, we also have, at each point $p \in U$

$$H_k^{l/k}(p) \geq H_l(p) \geq C_k H_k(p) > 0.$$

Hence $H_k^{(l-k)/k}(p) \geq C_k > 0$ on U, which gives

$$H_l(p) \geq C_k C_k^{k/(l-k)} = C_k^{l/(l-k)} > 0,$$

showing that $U = \{p \in M : H_l(p) \geq C_k^{l/(l-k)} > 0\}$ is also closed. Therefore $M = U$ and $H_l > 0$ on the whole M, as we claimed. Now, we have, from the Minkowski formula

$$\int_M H_{l-1}\langle a, \psi\rangle \, \mathrm{d}M = \int_M H_l \langle a, \eta\rangle \, \mathrm{d}M.$$

On the other hand, we have from the assumption of the theorem and the Minkowski formula

$$\begin{aligned}
&\int_M H_l\langle a, \eta\rangle \, \mathrm{d}M \\
&\quad = C_1 \int_M H_1\langle a, \eta\rangle \, \mathrm{d}M + C_2 \int_M H_2\langle a, \eta\rangle \, \mathrm{d}M + \cdots + C_{l-1} \int_M H_{l-1}\langle a, \eta\rangle \, \mathrm{d}M \\
&\quad = C_1 \int_M \langle a, \psi\rangle \, \mathrm{d}M + C_2 \int_M H_1\langle a, \psi\rangle \, \mathrm{d}M + \cdots + C_{l-1} \int_M H_{l-2}\langle a, \psi\rangle \, \mathrm{d}M.
\end{aligned}$$

Then it follows that:

$$\int_M \left(H_{l-1} - \sum_{i=1}^{l-1} C_i H_{i-1} \right) \langle a, \psi\rangle \, \mathrm{d}M = 0.$$

Since both $\langle a, \psi\rangle$ and $H_{i-1} - \sum_{i=1}^{l-1} C_i H_{i-1}$ do not change sign on M^n, it then follows that

$$H_{l-1} - \sum_{i=1}^{l-1} C_i H_{i-1} \equiv 0$$

on M^n. Then by Lemma 2, every point should be an umbilical point, that is, M^n is a totally umbilical round sphere.

Acknowledgements

Sung-Eun Koh was supported by Konkuk University in 2002 and Myung-Suk Yoo was supported by the KOSEF R14-2002-007-01003-0. This work was done while Sung-Eun Koh was visiting the Department of Mathematics, University of Michigan. He would like to thank the institution for their hospitality. The authors would also like to thank the referee for pointing out some mistakes and for valuable suggestions yielding a stronger and better result.

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